2.5-D Modeling in electromagnetic methods of geophysics

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Abstract

Understanding, using, or eliminating three-dimensional (3-D) effects in electromagnetic methods of geophysics are critical requirements. Numerous achievements in 3-D modeling sometimes give the impression that they are widely available today in geophysical practice. This is not necessarily true. Existing 3-D modeling packages prove that we know how to perform 3-D modeling. However, the computer resources and costs involved make the practical application of 3-D EM modeling in geophysical applications very limited.

A practical compromise, or even alternative, is represented by 2.5-D modeling characterized by the use of a 3-D source in a 2-D medium. This combination allows one to mathematically describe the related boundary value problem as a sequence of independent two-dimensional problems. The typical technique leading to such a split formulation is Fourier analysis. That is why the individual terms of a split solution are often referred to as harmonics.

Although each independent problem is two-dimensional, the algorithmic implementation of finite differences or integral equations for the higher harmonics has some specific features not present in the classical 2-D cases. In this paper, a hybrid scheme consisting of a combination of the finite difference technique with the integral equation approach for transient fields is described. Evaluation of algorithm accuracy is presented and a transient logging technique application is considered. The algorithm is fast and easily implemented on a personal computer.

1. Introduction

In different geophysical applications, state-of-the-art electromagnetic inversion algorithms generally require detailed simulation of transmitter–receiver configurations in a 3-D environment. In particular, we mention primarily the surface prospecting services (Strack, 1992; Goldman et al., 1994), logging (Gianzero et al., 1990), and airborne (Macnae et al., 1991). The majority of inversion schemes are based on an extensive repetitive modeling of the earth responses. The computer resources required for 3-D electromagnetic simulation severely restrict the practical applicability of any automated interpretation technology. The models of fractional dimensionality introduced by Dmitriev in 1969 suggested that progress in three-dimensional modeling for geophysics could be expected not only from a direct development of 3-D algorithms but also from a reasonable compromise between the degree of geophysically required model complexity and the level of computer

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resources consumption. It is not necessary that geophysical interpretations dictate the highest model complexity. If we analyze models in the sense of EM resolution, we can conclude that, in the majority of cases, we do not need the sharp boundaries or as many layers as we may have been inclined to take into account for an ideal analysis. This means that in mathematical modeling for geophysics, we have the invaluable resource of simplifying the model to a level corresponding to the physical limits of geophysical array resolution. In many cases, this procedure leads to ideal models (axisymmetric or two dimensional) in the presence of real 3-dimensional transmitters — a situation that is widely called 2.5-D. The major algorithmic advantage of 2.5-D models is that 3-D situations can be modeled as a sequence of 2-D models, resulting in a significant reduction in computer resources compared with 3-D algorithms.

In this paper, we consider the mathematical theory of 2.5-D transient EM modeling in an axially symmetric medium. One of the major factors complicating the algorithm is the presence of a nonconductive area in the model (upper half-space). The field satisfies the Laplace equation in this area, while in a conductive earth (lower half-space), the field is governed by a heat transfer equation. In the lower half-space, we can take advantage of the alternating directions method (Ames, 1977); however, if we use a uniform approach to solve the problem in both conductive and non-conductive areas, a demanding iterative process is required at any given time to satisfy boundary conditions at the earth surface.

In order to avoid the difficulty just described, a hybrid scheme is considered to allow for eliminating non-conductive areas from the modeling scheme. An integral equation arising from Green’s formula over the conductive/non-conductive area interface leads to a closed boundary value formulation exclusive to conductive regions. This approach was suggested independently by Weidelt (1975) and Tabarovsky and Krivoputsky (1978). After deriving this approach, we verify the developed program by comparison with several known results (including 3-D modeling), and also through a reciprocity check. Finally, a transient EM downhole logging tool is modeled.

2. Model symmetry and mathematical dimensionality of the boundary value problems

The physical dimensionality of a geological problem does not necessarily coincide with the mathematical dimensionality of the appropriate geophysical problem. The mathematical dimensionality can be reduced by choosing a proper geophysical model that possesses a certain spatial symmetry. Because of symmetry, the solution is independent of one of the spatial coordinates. As a result, the appropriate derivatives in the governing differential equation vanish, thus causing the reduction in mathematical dimensionality of the appropriate boundary value problem.

A classic example is a spherical body in a homogeneous space. This problem is treated mathematically as a 1-D boundary value problem (BVP) in which the model parameters depend only on radial coordinates in a spherical coordinate system. At the same time, this model may describe some simple 3-D geological targets such as isometric ore bodies, etc. In more realistic cases, when the earth–air interface is included in the model and the primary field is arbitrary, the appropriate BVP is generally three-dimensional. However, in some cases, BVPs for geological 3-D models can be reduced to a sequence of much simpler mathematical 2-D problems. Such problems were first introduced by Dmitriev (1969) who called them “quasi-three-dimensional” or Q3D problems.

Dmitriev’s model includes a 2-D structure within a horizontally layered earth in the presence of an arbitrary 3-D primary field. By applying the Fourier transform along the strike direction, the BVP is reduced to a sequence of 2-D problems.

A significant contribution to evaluating Q3D problems was made by Hohmann (1987) who referred to them as 2.5-D. Note that the difference in Dmitriev’s and Hohmann’s definition is not merely semantic: the use of the fractional index to designate the dimensionality of BVP permits a clearer emphasis of the different roles played
by both the model and the primary field dimensionality in the general dimensionality of the BVP. The decisive factor is the dimensionality of the medium and, therefore, it is specified by the integer part of the index. This also reflects the fact that the dimensionality of the BVP cannot be less than the dimensionality of the model.

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parameters. The presence of a fractional part in the dimensionality index shows that the dimensionality of the primary field is greater than that of the model parameters.

If the dimensionality of the primary field is less than or equal to that of the model parameters, the BVP dimensionality is expressed by an integer index. Such an approach allows us to classify geoelectric problems with respect to their dimensionality. A classification summary ranging from 1-D to 3-D is given in Table 1. For example, prior classifications make no provision for the fractional dimensionality in the case of 1-D model parameters and a multidimensional primary field. According to the proposed classification, all problems that involve 1-D models are referred to as 1-D problems, although mathematical dimensionality of the appropriate BVPs and, as a result, the numerical complexity of the algorithms are quite different for different primary excitations.

The important feature of the proposed classification is that it groups essentially different geophysical problems according to their computational similarity. Indeed, prior classifications reserve the term "two-dimensional" for infinitely long structures only. The proposed classification also includes axially symmetric structures in this class of models. In both cases, the physical properties of the models depend on only two orthogonal coordinates and, therefore, the appropriate numerical solutions are similar. Such axially symmetric models can be used in many cases to model 3-D geological objects with no less confidence but with much greater computational efficiency than is traditionally achieved using regular 3-D bodies.

Another important feature of the proposed classification is that it adequately indicates the computational complexity of the appropriate BVP. For example, the 2.5-D problem is a sequence of 2-D problems and, thus, is more complicated than a 2-D problem. Using the computational complexity approach, the proposed classification can include some further subdivisions. Tabarovsky et al. (1988) considered an axially symmetric structure with perturbed boundaries in which, by the perturbation method, he then reduced the problem to a sequence of 2.5-D problems. As a result, the algorithm was more complicated than a single 2.5-D algorithm, but still essentially simpler than that for the complete 3-D problem. Tabarovsky et al. (1988) referred to the problem as a 2.75-D problem.

3. 2.5-D Model geometry

Fig. 1 shows a conductive half-space, subdivided by coaxial cylinders and horizontal planes into a number of cylindrical cells with different conductivities $\sigma_k$. Transmitter Tx and receiver Rx can be positioned arbitrarily.
The source is characterized by the distribution of extraneous currents $j^s(r, \varphi, z, t)$. The axis $z$ of the cylindrical coordinate system $(r, \varphi, z)$ coincides with the cylindrical cell axis of symmetry.

4. Differential equations

A solution is sought for the electric vector potential, defined as follows:

$$H = \nabla \times A$$

$$E = -\mu \frac{\partial A}{\partial t} + \nabla \cdot \frac{\nabla \cdot A}{\sigma}$$

where $\sigma$ and $\mu$ are conductivity and magnetic permeability at a given point (assume that $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m everywhere); $H$, $E$, and $A$ are vectors of magnetic and electric fields and the vector potential, respectively. It is useful to note that Eq. (2) was obtained using the Lorenz condition, i.e.:

$$\nabla \cdot A + \sigma \varphi = 0$$

where $\varphi$ is the scalar potential. Using the Maxwell equations, we obtain:

$$\nabla^2 A = \mu \sigma \frac{\partial A}{\partial t} + j^s$$

where $j^s$ is the extraneous current density. In order to simplify Eq. (3) and its finite difference (FD) algorithm, a solution is sought either for the secondary vector potential (within the domain where the extraneous currents are located) or for the total vector potential (in the remaining space). The secondary potential $A^s$ is defined as the difference between the total potential $A$ and the primary potential $A^p$:

$$A^s = A - A^p$$

where the primary potential $A^p$ is chosen as a solution of Maxwell equations in a homogeneous space having the conductivity of the domain enclosing the source. This procedure complicates the boundary conditions slightly, but greatly simplifies Eq. (3) and, as a result, the FD algorithm in the vicinity of the source (Tabarovsky and Goldman, 1978; Goldman and Stoyer, 1983; Hohmann, 1987). Eq. (3) for the vector potential being sought then becomes homogeneous:

$$\nabla^2 A = \mu \sigma \frac{\partial A}{\partial t}$$

Expanding the potential in a Fourier series with angular harmonics in the form:

$$A = \sum_{n=0}^{\infty} A^{n,c} \cos n \varphi + A^{n,i} \sin n \varphi$$

and substituting this result into Eq. (4), using a cylindrical coordinate system, two independent systems of equations for the Fourier amplitudes of vector $A$ can be obtained:

$$\nabla^2 A^{n,c} = \mu \sigma \frac{\partial A^{n,c}}{\partial t}$$

$$\nabla^2 A^{n,i} = \mu \sigma \frac{\partial A^{n,i}}{\partial t}$$

$$\nabla^2 A^{n,c} = \mu \sigma \frac{\partial A^{n,c}}{\partial t}$$

$$\nabla^2 A^{n,i} = \mu \sigma \frac{\partial A^{n,i}}{\partial t}$$

$$\nabla^2 A^{n,c} = \mu \sigma \frac{\partial A^{n,c}}{\partial t}$$

$$\nabla^2 A^{n,i} = \mu \sigma \frac{\partial A^{n,i}}{\partial t}$$

$$\nabla^2 A^{n,c} = \mu \sigma \frac{\partial A^{n,c}}{\partial t}$$

$$\nabla^2 A^{n,i} = \mu \sigma \frac{\partial A^{n,i}}{\partial t}$$
in which:

\[ V_\varphi^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} + \frac{\partial^2}{\partial z^2} \]  

(8)

It may be seen that the difference between the two systems is that the first combines the \( A_\varphi^{n,c} \), \( A_\psi^{n,c} \), and \( A_n^{n,c} \) components and the second combines the \( A_\varphi^{n,c} \), \( A_\psi^{n,c} \), and \( A_n^{n,c} \) components. According to the succession of the indices, we shall refer to the former combination of components as CSC-mode and to the latter as SCS-mode. The presence of either mode in the solution is determined entirely by the type and geometry of the source. The most remarkable feature of the solution is that by choosing a proper coordinate system, the solution can be, as a rule, reduced to a single mode. The coupled expressions in Eqs. (6) and (7) are next transformed into independent systems of equations. Let us consider, for example, the SCS mode. We introduce new functions \( B^n \) and \( C^n \) of the form:

\[ B^n = A_\varphi^{n,1} + A_\psi^{n,c} \]  

(9)

\[ C^n = A_\varphi^{n,1} - A_\psi^{n,c} \]  

(10)

After adding and subtracting the first two equations, the system in Eq. (7) splits into the following three independent equations:

\[ V_\varphi^2 B^n - \frac{2n}{r^2} B^n = \mu \sigma \frac{\partial B^n}{\partial t} \]  

(11)

\[ V_\varphi^2 C^n - \frac{2n}{r^2} C^n = \mu \sigma \frac{\partial C^n}{\partial t} \]  

(12)

\[ V_\varphi^2 A_\varphi^n = \mu \sigma \frac{\partial A_\varphi^n}{\partial t} \]  

(13)

A comparison of Eqs. (7) and (6) reveals that they differ only in the signs of the terms of proportional \( n \). Consequently, the equations in (6) can be separated in exactly the same manner as equations in (7) with the resulting system differing from Eqs. (11)–(13) only in the signs of the terms of proportional \( n \). The FD algorithm considered in the later evaluations is designed for the system in Eqs. (11)–(13).

5. Boundary conditions

Taking into account the continuity of the tangential field components, the following boundary conditions for the potentials are valid:
Horizontal boundary:

\[ [B^*] = 0 \left[ \frac{\partial B^*}{\partial z} \right] = 0 \]  \hspace{1cm} (14a)

\[ [C^*] = 0 \left[ \frac{\partial C^*}{\partial z} \right] = 0 \]  \hspace{1cm} (14b)

\[ [A^*_r] = 0 \left[ \frac{\nabla A^*}{\sigma} \right] \]  \hspace{1cm} (14c)

Vertical boundary:

\[ [A^*_r] = 0 \left[ \frac{\partial A^*_r}{\partial r} \right] = 0 \]  \hspace{1cm} (15a)

\[ [B^*] = 0 \left[ C^* \right] = 0 \]  \hspace{1cm} (15b)

\[ \left[ \frac{\partial (B^* - C^*)}{\partial r} \right] = 0 \left[ \frac{\nabla A^*}{\sigma} \right] = 0 \]  \hspace{1cm} (15c)

where

\[ \nabla A^* = \frac{1}{r} \left( B^* + C^* \right) + \frac{\partial}{\partial r} \left( B^* + C^* \right) - \frac{n}{r} \frac{B^* - C^*}{2} + \frac{\partial A^*_r}{\partial z} \]  \hspace{1cm} (16)

The symbol \([\ ]\) in Eqs. 14a–14c and 15a–15c means the difference between the values of the specified functions on opposite sides of the boundary (e.g., \([A] = 0\) means that potential \(A\) is continuous across the boundary).

An examination of the boundary conditions (14) through (16) shows that some of them are coupled (e.g., condition \([\nabla A^*/\sigma] = 0\) connects all three potentials sought). This is the most complicated aspect of the FD algorithm being considered. The problem is solved by choosing the proper succession in calculating the potentials and by using an alternating direction-implicit (ADI) scheme for the solution to the FD problem (Ames, 1977). This question is discussed in greater detail later in the section entitled “Step-by-step algorithm description”.

In order to formulate the boundary-value problem in closed form, we need additional conditions at infinity, at the axis of symmetry, and at the earth–air interface. The conditions at infinitely remote points are trivial and follow directly from physical considerations:

\[ B^* \rightarrow 0 \]  \hspace{1cm} (17)

\[ C^* \rightarrow 0 \quad z \rightarrow \pm \infty; \quad r \rightarrow \infty \]  \hspace{1cm} (18)

\[ A^*_r \rightarrow 0 \]  \hspace{1cm} (19)

Special consideration must be given to the boundary conditions on the axis of symmetry \((r = 0)\) to close the solution in the radial direction and to apply the ADI algorithm in the vertical direction on the axis of symmetry. It is not possible to use the previously obtained differential equations (Eqs. (11), (12) and (13)) on the axis of symmetry because these equations contain terms with \(1/r\). Therefore, we have to change the radial operator in these equations by using the symmetry of the field harmonics. For this purpose, it is clear that the vector
Fig. 2. Grid nodes in the neighborhood of the symmetry axis.

The potential has no peculiarities on the axis of symmetry, which means that all three components, \( A_z \), \( A_r \), and \( A_\phi \), exist. Using the Fourier harmonics representation, it can be proven that only the first harmonics of \( A_r \) and \( A_\phi \) and the zero harmonic of \( A_z \) exist on the axis of symmetry. Thus, we can conclude that:

\[
\begin{align*}
B^n &= 0 \quad n = 0,2,3,\ldots \\
C^n &= 0 \quad n = 0,2,3,\ldots \\
A^z_n &= 0 \quad n = 1,2,3,\ldots
\end{align*}
\]

(20)

Let us begin with the zero harmonics of the \( A_z \) component. We can represent the radial operator in Eq. (13) in finite-difference form using symbols from Fig. 2:

\[
\mu \sigma \frac{\partial A^0_z}{\partial t} = \frac{\partial^2 A^0_z}{\partial z^2} + \frac{A^0_z(1) + A^0_z(2) + A^0_z(3) + A^0_z(4) - 4A^0_z(5)}{\Delta r^2}
\]

(21)

Taking into account the axial symmetry of the solution for the zero harmonic (for harmonic \( n = 0 \), the solution does not depend on the angle \( \phi \), which means that \( A^0_z(1) = A^0_z(2) = A^0_z(3) = A^0_z(4) \)), we can obtain the necessary condition for \( A^0_z \) from:

\[
\frac{4(A^0_z(\Delta r) - 2A^0_z(0))}{\Delta r^2} + \frac{\partial^2 A^0_z}{\partial z^2} = \mu \sigma \frac{\partial A^0_z}{\partial t}
\]

(22)

As mentioned previously, the vector potentials have no peculiarities on the axis of symmetry and, thus, we can write them either in Cartesian coordinates (Fig. 3) as:

\[
A = (A_z, A_r, A_\phi) = (a \cos \varphi_0, a \sin \varphi_0, A_z)
\]

(23)

or in cylindrical coordinates as:

\[
A_z = a \cos (\varphi - \varphi_0) = (a \cos \varphi_0) \cos \varphi + (a \sin \varphi_0) \sin \varphi
\]

\[
A_\phi = -a \sin (\varphi - \varphi_0) = (-a \cos \varphi_0) \sin \varphi + (a \sin \varphi_0) \cos \varphi
\]

(24)

The two expressions in Eq. (24) prove that only the first harmonics of \( A_r \) and \( A_\phi \) components exist at the axis of symmetry. For the SCS mode, we can obtain the following expressions from Eq. (24):

\[
\begin{align*}
A^z_{r,1} &= a \sin \varphi_0 \\
A^z_{\phi,1} &= a \sin \varphi_0
\end{align*}
\]

(24a)
Using variables $B$ and $C$, we can derive:

**SCS mode**

$$B^1 = 2a \sin \varphi_0$$

$$C^1 = 0$$

(25)

**CSC mode**

$$B^1 = 0$$

$$C^1 = 2a \cos \varphi_0$$

(26)

Comparing Eqs. (25) and (26) with Eq. (23), we note that for the SCS mode $B^1 = 2 A^0$, and for the CSC mode $C^1 = 2 A^0$.

Next, we will prove that components $A^0_0$ and $A^0_1$ satisfy equations identical to Eq. (22) for $A^0_2$ on the axis of symmetry.

1. In the Cartesian coordinate system, the potential $A_2(x, y, z)$ obviously satisfies the heat transfer equation:

$$\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_2}{\partial z^2} = \mu \sigma \frac{\partial A_2}{\partial t}$$

(27)

We can rewrite this equation in finite difference form with respect to the coordinates $x, y$ explicitly using the cylindrical coordinates of the points 1, 2, 3, 4 indicated in Fig. 2. That is:

$$A_2(\Delta r, 0) + A_2(\Delta r, \pi) + A_2(\Delta r, \pi) + A_2(\Delta r, 3\pi) - 4A_2(0, 0)$$

$$\frac{\partial^2 A_2}{\partial z^2} = \mu \sigma \frac{\partial A_2}{\partial t}$$

$$\Delta r^2$$
(2) Next, calculate the terms in the finite difference part of Eq. (27). First, we present the Fourier expansion of the function \( A_z \) as:

\[
A_z(\Delta r, \varphi) = \sum_{k=0}^{\infty} A_z^{k,c} \cos(k\varphi) + \sum_{k=1}^{\infty} A_z^{k,s} \sin(k\varphi)
\]  

(28)

Taking into account the succession of signs produced by the functions \( \sin(n\varphi) \) and \( \cos(n\varphi) \) at the points 1-4 (Fig. 2) while \( n \) is changing, we can derive the following equations from Eq. (28):

\[
A_z(\Delta r, 0) = A_z^{0,c} + A_z^{1,c} + A_z^{2,c} + A_z^{3,c} + A_z^{4,c} + \ldots
\]

\[
A_z(\Delta r, \pi/2) = A_z^{0,c} + A_z^{1,s} - A_z^{2,c} - A_z^{3,c} + A_z^{4,c} + \ldots
\]

\[
A_z(\Delta r, \pi) = A_z^{0,c} - A_z^{1,c} + A_z^{2,c} - A_z^{3,c} + A_z^{4,c} + \ldots
\]

\[
A_z(\Delta r, 3\pi/2) = A_z^{0,c} - A_z^{1,s} - A_z^{2,c} + A_z^{3,s} + A_z^{4,c} + \ldots
\]

After summation of these equations, we obtain:

\[
\frac{4(A_z^{0,c}(\Delta r) - A_z^{0,c}(\Delta r = 0))}{\Delta r^2} + \frac{4A_z^{4,c}}{\Delta r^2} + \frac{\partial^2 A_z}{\partial z^2} = \mu \sigma \frac{\partial A_z}{\partial t}
\]

(29)

(3) Next, we shall prove that the amplitude of the \( n \)th harmonic \( A_z^{n,c} \) is of order \( (\Delta r)^n \). To accomplish this, we expand the component \( A_z \) in a Taylor series as a function of \( \Delta x \) and \( \Delta y \) to get:

\[
A_z(\Delta x, \Delta y) = \sum_{k=1}^{\infty} \frac{\partial^k A_z}{\partial x^k} \Delta x^k + \frac{\partial^k A_z}{\partial y^k} \Delta y^k = \sum_{k=0}^{\infty} \frac{\partial^k A_z}{\partial x^k} (\Delta r \cos \varphi)^k + \frac{\partial^k A_z}{\partial y^k} (\Delta r \sin \varphi)^k
\]

\[
= \sum_{k=0}^{\infty} \frac{\partial^k A_z}{\partial x^k} \Delta r^k \left( \frac{e^{i\varphi} + e^{-i\varphi}}{2} \right)^k + \frac{\partial^k A_z}{\partial y^k} \Delta r^k \left( \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \right)^k
\]

(30)

All derivatives in Eq. (30) must be taken at the center of the model \((x = 0, y = 0)\). One can easily conclude from Eq. (30) that the terms containing \( \cos(n\varphi) \) or \( \sin(n\varphi) \) first appear in the sum when \( k = n \), indicating that the \( n \)th harmonic is of order \( (\Delta r)^n \). Thus, the term \( (4A_z^{n,c}/(\Delta r)^2) \) in Eq. (29) is of order \( (\Delta r)^2 \) and consequently can be omitted. Finally, we obtain the following equations for \( B^1 \), \( C^1 \) on the model axis:

**SCS mode**

\[
\frac{4(B^1(\Delta r) - B^1(0))}{\Delta r^2} + \frac{\partial^2 B^1}{\partial z^2} = \mu \sigma \frac{\partial B^1}{\partial t}
\]

\( C^1 = 0 \)

**CSC mode**

\[
B^1 = 0
\]

\[
\frac{4(C^1(\Delta r) - C^1(0))}{\Delta r^2} + \frac{\partial^2 C^1}{\partial z^2} = \mu \sigma \frac{\partial C^1}{\partial t}
\]

6. Earth–air interface

In order to avoid solving a completely different (elliptical) equation in the air half-space, a special boundary condition at the earth’s surface is needed. This was formulated by Tabarovsky and Krivoputsky (1978) and
Tabarovsky and Goldman (1978) for a 2-D axially symmetric problem and by Oristaglio and Hohmann (1984) for a 2-D generalized problem. The equations for the vector potentials of 2.5-D problems were fully formulated by Tabarovsky (1982).

Following this procedure, we can obtain the boundary conditions that are necessary for our algorithm. The presence of an insulated space \( V_i(\sigma_i = 0) \) changes the Lorenz condition (Eq. (2a)) to:

\[
\nabla \cdot \mathbf{A} = 0
\]

and the boundary condition between the conductive and insulated regions for \( \text{div} \, \mathbf{A} \) from the insulated side becomes:

\[ (\nabla \cdot \mathbf{A})_{\text{ins}} = 0 \tag{31} \]

where the subscript \( \text{ins} \) indicates the insulated side.

Assuming that the insulated region \( V_i \) is bounded by a plane \( S \), the vector potentials in the region \( V_i \) are defined using the potential normal derivatives along the direction \( n \) perpendicular to the plane \( S \) and the Green function \( G \) with \( \partial G / \partial n = 0 \) on \( S \):

\[
\mathbf{A} = \int_S G \left( \begin{array}{c} \frac{\partial A_1}{\partial n} \\ \frac{\partial A_2}{\partial n} \\ - \nabla \cdot \mathbf{A}_S \end{array} \right) dS
\]

Taking Eq. (31) into account, we obtain:

\[
\mathbf{A} = \int_S G \left( \begin{array}{c} \frac{\partial A_1}{\partial n} \\ \frac{\partial A_2}{\partial n} \\ - \nabla \cdot \mathbf{A}_S \end{array} \right) dS \tag{32}
\]

in which the surface divergence of vector potential \( \mathbf{A} \) is given by \( \nabla \cdot \mathbf{A}_S \). In deriving Eq. (32), we have assumed that the normal vector \( n \) is directed from insulator to conductor. \( A_1 \) and \( A_2 \) are the tangential components of the vector potential. In the case of a plane boundary, the Green’s function in the Cartesian coordinate system \((x,y,z)\) is described as:

\[
G = \frac{1}{2\pi \rho} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \tag{33}
\]

where \( \rho = [(x-x_0)^2 + (y-y_0)^2]^{1/2} = [r^2 + r_0^2 - 2rr_0 \cos (\varphi - \varphi_0)]^{1/2} \) \((x_0,y_0)\) \((x,y)\) or \((Q_0,\varphi_0)\), \((\rho,\varphi)\) are coordinates at points \( Q_0 \) and \( Q \) located on the plane. Using cylindrical coordinates in Eq. (32):

\[
\begin{align*}
(A_r) &= \int_S \frac{1}{\rho} \left( \begin{array}{ccc} \cos (\varphi - \varphi_0) & -\sin (\varphi - \varphi_0) & 0 \\ \sin (\varphi - \varphi_0) & \cos (\varphi - \varphi_0) & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{c} \frac{\partial A_r}{\partial n} \\ \frac{\partial A_\varphi}{\partial n} \\ - \nabla \cdot \mathbf{A}_S \end{array} \right) dS \\
(A_\varphi) &= \int_S \frac{1}{\rho} \left( \begin{array}{ccc} \cos (\varphi - \varphi_0) & -\sin (\varphi - \varphi_0) & 0 \\ \sin (\varphi - \varphi_0) & \cos (\varphi - \varphi_0) & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{c} \frac{\partial A_\varphi}{\partial n} \\ \frac{\partial A_z}{\partial n} \\ - \nabla \cdot \mathbf{A}_S \end{array} \right) dS \tag{34}
\end{align*}
\]

Expanding the vector potential \((A_r,A_\varphi,A_z)\) and function \(1/\varrho\) in Fourier series yields:

\[
\begin{align*}
(A_r) &= \sum_{n=0}^{\infty} \left( \begin{array}{cc} \cos n\varphi & 0 \\ 0 & \sin n\varphi \\ 0 & \cos n\varphi \end{array} \right) \left( \begin{array}{c} A_r^{n,r} \\ A_\varphi^{n,\varphi} \\ A_z^{n,z} \end{array} \right) + \sum_{n=0}^{\infty} \left( \begin{array}{cc} \sin n\varphi & 0 \\ 0 & \cos n\varphi \\ 0 & \sin n\varphi \end{array} \right) \left( \begin{array}{c} A_r^{n,\varphi} \\ A_\varphi^{n,\varphi} \\ A_z^{n,\varphi} \end{array} \right) \tag{35}
\end{align*}
\]
\[
\frac{1}{Q} = \sum_{n=-\infty}^{\infty} g_n \cos n(\varphi - \varphi_0)
\]  

(36)

Substituting Eqs. (35) and (36) into Eq. (34) and integrating with respect to \( \varphi \), we can obtain:

\[
\begin{pmatrix}
A_r \\
A_\varphi \\
A_z
\end{pmatrix}
= \sum_{n=0}^{\infty} \begin{pmatrix}
\cos n\varphi & 0 & 0 \\
0 & \sin n\varphi & 0 \\
0 & 0 & \cos n\varphi
\end{pmatrix}
\int_0^\infty \begin{pmatrix}
\frac{g_{n-1} + g_{n+1}}{2} & -\frac{g_{n-1} - g_{n+1}}{2} & 0 \\
-\frac{g_{n-1} - g_{n+1}}{2} & \frac{g_{n-1} + g_{n+1}}{2} & 0 \\
\frac{g_{n-1} + g_{n+1}}{2} & 2 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial A_r^n}{\partial n} \\
\frac{\partial A_\varphi^n}{\partial n} \\
\frac{\partial A_z^n}{\partial n}
\end{pmatrix}
\mathrm{d}r \mathrm{d}r
\]

\[
+ \sum_{n=0}^{\infty} \begin{pmatrix}
\sin n\varphi & 0 & 0 \\
0 & \cos n\varphi & 0 \\
0 & 0 & \sin n\varphi
\end{pmatrix}
\int_0^\infty \begin{pmatrix}
\frac{g_{n-1} + g_{n+1}}{2} & 0 & 0 \\
0 & \frac{g_{n-1} + g_{n+1}}{2} & 2 \\
0 & 2 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial A_r^n}{\partial n} \\
\frac{\partial A_\varphi^n}{\partial n} \\
\frac{\partial A_z^n}{\partial n}
\end{pmatrix}
\mathrm{d}r \mathrm{d}r
\]

(37)

where

\[
D^{n,c} = -\left( \frac{1}{r} \frac{\partial}{\partial r}(rA_r^n) + \frac{n}{r} A_\varphi^n \right)
\]

\[
D^{n,i} = -\left( \frac{1}{r} \frac{\partial}{\partial r}(rA_\varphi^n) - \frac{n}{r} A_r^n \right)
\]

Substituting the variables \( B^n(9) \) and \( C^n(10) \) in Eq. (37) and taking into account two mode structures, we obtain the following equations:

**CSC mode**

\[
\begin{pmatrix}
B^n \\
C^n \\
A_z
\end{pmatrix}
= \int_0^\infty \begin{pmatrix}
g_{n+1} & 0 & 0 \\
0 & g_{n-1} & 0 \\
0 & 0 & g_n
\end{pmatrix}
\begin{pmatrix}
\frac{\partial B^n}{\partial n} \\
\frac{\partial C^n}{\partial n} \\
\frac{\partial A_z^n}{\partial n}
\end{pmatrix}
\mathrm{d}r \mathrm{d}r
\]

(38)

and

**SCS mode**

\[
\begin{pmatrix}
B^n \\
C^n \\
A_z
\end{pmatrix}
= \int_0^\infty \begin{pmatrix}
g_{n-1} & 0 & 0 \\
0 & g_{n+1} & 0 \\
0 & 0 & g_n
\end{pmatrix}
\begin{pmatrix}
\frac{\partial B^n}{\partial n} \\
\frac{\partial C^n}{\partial n} \\
\frac{\partial A_z^n}{\partial n}
\end{pmatrix}
\mathrm{d}r \mathrm{d}r
\]

(39)
It should be noted that Eqs. (38) and (39) are split into the same modes as both the input equation and the boundary conditions. These integral equations express potential values through their derivatives from the insulated side. However, from boundary conditions (Eq. (14)), \( \partial B^*/\partial z \) and \( \partial C^*/\partial z \) are continuous. This allows the use of corresponding values of potentials and derivatives from the conductive side of the boundary. Next, integral Eqs. (38) and (39) can be incorporated into the finite-difference scheme in order to close the solution in the vertical direction. Therefore, the finite-difference algorithm becomes hybrid in the sense that it also uses integral equations.

7. Initial conditions and primary fields

The initial conditions are determined by the structure of the extraneous currents. Distribution of the extraneous currents and their peculiarities are taken into account by the primary field, as mentioned previously. It is therefore expedient to describe the initial conditions and primary fields together. A total current \( J \) in the source of the primary field for transient processes can be described as a step function of time:

\[
J = \begin{cases} 
0 & t < 0 \\
I & t > 0
\end{cases} \quad \text{— switch-on}
\]

\[
J = \begin{cases} 
I & t < 0 \\
0 & t > 0
\end{cases} \quad \text{— switch-off}
\]

Since the solution for the secondary potentials is sought in the medium containing extraneous currents, the initial condition is zero for this domain. We will now discuss the initial conditions for a domain not containing extraneous currents. For the switch-on process, of course, the initial condition is zero throughout the entire space.

7.1. Dipole sources in a uniform conducting space

Expressions for the potentials of primary fields for corresponding dipoles are presented in Table 2. The symbols used in this table are:

- \( I \) — electric dipole
- \( M \) — magnetic dipole

\[
\Phi(u) = -\frac{2}{\sqrt{ \pi t } } \int_0^r e^{-\frac{x^2}{t}} \, dx
\]

\[
u = \frac{R}{\sqrt{4t/\mu\sigma}}
\]

\[
R = \left( r_0^2 + r^2 - 2rr_0 \cos(\varphi - \varphi_0) + (z - z_0)^2 \right)^{\frac{1}{2}}
\]

\( r_0, \varphi_0, z_0 \) — coordinates at the source

\( r, \varphi, z \) — coordinates at the point under consideration

DC symbols shown for the switch-off process for the electric dipoles mean that the initial conditions are equal to the values of corresponding solutions for a direct current source. The same values can be obtained as late-time asymptotic values for the switch-on process. The presence of either the CSC or the SCS mode in the solution (see Section 2.5-D Model geometry) depending on the type and geometry of the source, is also
Table 2
Primary fields

<table>
<thead>
<tr>
<th>Dipole</th>
<th>Mode</th>
<th>Vector potential</th>
<th>Initial condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Switch-off</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_1$</td>
<td>CSC</td>
<td>$A_x = 0$</td>
<td>$A_x = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = 0$</td>
<td>$A_y = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = \frac{\phi(u)}{4\pi R}$</td>
<td>$A_z = DC$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_x = \frac{\phi(u) \cos \varphi}{4\pi R}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = -\frac{\phi(u) \sin \varphi}{4\pi R}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = 0$</td>
<td></td>
</tr>
<tr>
<td>$I_2$</td>
<td>CSC</td>
<td>$A_x = 0$</td>
<td>$A_x = DC$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = \frac{\phi(u) \sin \varphi}{4\pi R}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = \frac{\phi(u) \cos \varphi}{4\pi R}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = 0$</td>
<td></td>
</tr>
<tr>
<td>$I_3$</td>
<td>SCS</td>
<td>$A_x = 0$</td>
<td>$A_x = DC$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = \frac{\phi(u)}{4\pi R^3}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = 0$</td>
<td></td>
</tr>
<tr>
<td>$M_1$</td>
<td>SCS</td>
<td>$A_x = -\frac{r_0 \sin \varphi}{4\pi R^3} \left( \phi(u) - \frac{2u}{\sqrt{\pi}} e^{-u^2} \right)$</td>
<td>$A_x = DC$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = \frac{(r - r_0 \cos \varphi)}{4\pi R^3} \left( \phi(u) - \frac{2u}{\sqrt{\pi}} e^{-u^2} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = 0$</td>
<td></td>
</tr>
<tr>
<td><strong>Switch-on</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_1$</td>
<td>CSC</td>
<td>$A_x = 0$</td>
<td>$A_x = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = 0$</td>
<td>$A_y = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = \frac{1 - \phi(u)}{4\pi R}$</td>
<td>$A_z = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_x = \frac{1 - \phi(u) \cos \varphi}{4\pi R}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = -\frac{(1 - \phi(u)) \sin \varphi}{4\pi R}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = 0$</td>
<td></td>
</tr>
<tr>
<td>$I_2$</td>
<td>CSC</td>
<td>$A_x = 0$</td>
<td>$A_x = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = \frac{\phi(u)}{4\pi R}$</td>
<td>$A_y = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = 0$</td>
<td>$A_z = 0$</td>
</tr>
<tr>
<td>$I_3$</td>
<td>SCS</td>
<td>$A_x = 0$</td>
<td>$A_x = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = \frac{\phi(u) \sin \varphi}{4\pi R}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = \frac{\phi(u) \cos \varphi}{4\pi R}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = 0$</td>
<td></td>
</tr>
<tr>
<td>$M_2$</td>
<td>SCS</td>
<td>$A_x = -\frac{r_0 \sin \varphi}{4\pi R^3} \left( (1 - \phi(u)) + \frac{2u}{\sqrt{\pi}} e^{-u^2} \right)$</td>
<td>$A_x = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_y = \frac{(r - r_0 \cos \varphi)}{4\pi R^3} \left( (1 - \phi(u)) + \frac{2u}{\sqrt{\pi}} e^{-u^2} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_z = 0$</td>
<td></td>
</tr>
</tbody>
</table>
indicated in this table. The calculation of corresponding harmonics of the primary potentials can be performed numerically using the following equations:

\[
A^{n,0} = \frac{1}{\pi} \int_{-\pi}^{\pi} A \cdot \cos(n\varphi) \, d\varphi
\]

\[
A^{n,1} = \frac{1}{\pi} \int_{-\pi}^{\pi} A \cdot \sin(n\varphi) \, d\varphi
\]

7.2. Vertical magnetic dipole on the surface of a uniform conducting half-space (switch-off)

In the coordinate system attached to the dipole axis:

\[
E_z = -\frac{\partial A_y}{\partial t}
\]

Correspondingly:

\[
A_y = -\int_0^t E_z \, dt
\]  \hspace{1cm} (40)

Let us use the known equation (Morozova and Tabarovsky, 1978):

\[
E_z = \frac{8\sqrt{2}\pi}{\sigma \pi^2 R^4} \int_{r}^{r'} \left( u_r e^{-2u'} + \int_0^\infty f(\lambda) d\lambda \right) \hspace{1cm} (41)
\]

where

\[
f(\lambda) = \lambda^4 e^{-2\lambda^2 r'^2} \left( 3 - 4\lambda^2 (r')^2 + 4\lambda^2 H' \right)^2 - \sqrt{2\pi} H' e^{i\lambda^2 r'^2} \psi(\lambda)
\]

\[
\psi(\lambda) = \left( -4\lambda^2 (r')^2 + 3 \cdot I_0 \lambda^2 (r')^2 \right) + \left( 4\lambda^2 (r')^2 - 1 \right) \cdot I_1 (\lambda^2 (r')^2)
\]

\[
R = \sqrt{r^2 + z^2} \hspace{0.5cm} u_r = \frac{r}{2\sqrt{2} t'} \hspace{0.5cm} r' = \frac{r}{R} \hspace{0.5cm} H' = \frac{z}{R} \hspace{0.5cm} t^* = \frac{t}{\mu \sigma}
\]

\(r, z\) — coordinates of a receiver;
\(
\sigma
\) — conductivity of the half-space;
\(I_0, I_1\) — modified Bessel functions.

Substituting Eq. (41) into Eq. (40) and carrying out the integration, we obtain:

\[
A_y = \frac{8\sqrt{2}\pi}{\sigma \pi^2 R^4} \left( \frac{\sqrt{\pi} R^2}{32\sqrt{2}} \cdot \left( 1 - \Phi(u') \right) + \frac{2u'}{\sqrt{\pi}} e^{-u'^2} \right) + \int_0^\infty Kf(\lambda) d\lambda \hspace{1cm} (42)
\]

where

\[
u' = \frac{R}{2\sqrt{2} t'}
\]

\[
K = t^* \cdot \lambda < \frac{R}{2\sqrt{2} t'}
\]

\[
K = \frac{r^2}{8\lambda^2} \cdot \lambda > \frac{R}{2\sqrt{2} t'}
\]
The primary field components of the magnetic dipole displaced from the center of coordinates to the point \((r_0,0,0)\) can be calculated according to the following equations:

\[
A^\text{dis}_r = \frac{r_0 \sin \varphi}{r} \cdot A_r
\]
\[
A^\text{dis}_\varphi = \frac{r - r_0 \cos \varphi}{r} \cdot A_\varphi
\]

where

- \(A^\text{dis}_r, A^\text{dis}_\varphi\) — potentials of the displaced dipole;
- \(A_r\) — potential from the Eq. (41);
- \(r, \varphi, z\) — coordinates of a receiver.

Initial conditions are the same as for a magnetic dipole in unbounded space and only the SCS mode exists in the solution.

8. Step-by-step algorithm description

In order to fully understand the algorithm functioning as a whole, we present a step-by-step description of the algorithm action. It should be noted that the algorithm is based on an alternating direction-implicit (ADI) scheme for solving finite-difference 2-D problems (Peaceman and Rachford, 1955). Familiarity with the ADI scheme is necessary to understand the following description. The finite-difference formulation for 2-D problems, including incorporation in the solution of the boundary conditions and earth–air interface integral equation, is described in full by Tabarovskiy and Krivoputsky (1978), Tabarovskiy and Goldman (1978), and Goldman and Stoye (1983). However, all of these solutions deal with scalar problems and, in our case, we have three components in the solution that are coupled at every boundary by the condition \([\nabla A^n/\sigma] = 0\). In order to separate the solution, i.e., make three almost-independent scalar problems from one vector problem, we must apply a very specific sequence of steps in calculating the potentials. We shall show this sequence in an arbitrary time step of the FD solution.

8.1. First half-step: FD solution in radial direction for \(t + \Delta t/2\)

Examination of vertical boundary conditions (Eqs. 15) shows that we have two independent conditions for the \(A^\sigma_r\) component in Eqs. (15a). Incorporating these conditions in the FD formulation using a three-point derivative and solving tridiagonal linear systems, we obtain a solution at the intermediate time \(t + \Delta t/2\) for the \(A^\sigma_r\) component. After that, we can calculate the value \(\partial A^n_\varphi/\partial z\), which is necessary for the boundary condition \([\nabla A^n_\varphi/\sigma] = 0\) in Eq. (15c).

![Grid nodes of the radial grid near a vertical boundary.](image)
Now, Eqs. (15b) and (15c) depend only on $B^*$ and $C^*$ potentials, but they are coupled. In order to satisfy these boundary conditions and simultaneously take advantage of decoupled Eqs. (11)-(13), we perform runs from the left and right edges of the radial grid to the vertical boundary (Fig. 4). We then solve the system of Eqs. (15b) and (15c) with known value $\partial A^*_\gamma / \partial z$ and perform a back run from the vertical boundary to the edges.

8.2. Second half-step: FD solution in the vertical direction for $t + \Delta t$

Since horizontal boundary conditions in Eqs. (14a) and (14b) are separated for potentials $B^*$ and $C^*$, we can apply the same procedure for these potentials in the vertical direction as used for $A_\gamma$ in the radial direction. If we have the earth–air interface in the model, we can incorporate our scheme into Eq. (38) or Eq. (39), both of which are also decoupled for $B^*$ and $C^*$. To achieve this, we must begin the direct vertical run from the bottom of the grid. Using the coefficients of the direct vertical run, we can solve this Fredholm integral equation (Tabarovsky and Krivoputsky, 1978; Goldman and Stoyer, 1983) numerically to obtain values of $B^*$ and $C^*$ on the surface. These values are then used for initiating the back vertical run that produces a solution $B^*$ and $C^*$ in the entire area at the time $t + \Delta t$. Now, the differential equation for $A_\gamma$ can readily be solved. Using $B^*$ and $C^*$, we can calculate the auxiliary quantity (see Eq. (16)):

$$ q^* = \frac{1}{2} \left( \frac{B^* + C^*}{r} \right) + \frac{\partial}{\partial r} \left( \frac{B^* + C^*}{2r} \right) - \frac{n}{r} \left( \frac{B^* - C^*}{2r} \right) $$

Using this quantity, the boundary conditions (14c) for $A^*_\gamma$ potential become:

$$ \left[ A^*_\gamma \right] = 0 $$

$$ \left[ \frac{1}{\sigma} \left( q^* + \frac{\partial A^*_\gamma}{\partial z} \right) \right] = 0 $$

(45)

This result means that we have a separated BVP for potential $A_\gamma$. The procedure for solving this problem is exactly the same as for $B^*$ and $C^*$.

8.3. Nonequidistant spatial and temporal grids

Previous experience with FD calculations in 2-D models has shown that there are some disadvantages in applying equidistant spatial and temporal grids (Tabarovsky and Krivoputsky, 1978; Tabarovsky and Goldman, 1978). First of all, since the EM field variations decrease with time, the essential changes in the field in the late time stage of the process requires the calculation of many time steps. Secondly, the currents propagate to the edges of the modeling domain with increasing time and, correspondingly, the accuracy of the modeling results at the late time stage depends on the size of the modeled volume of the spatial grid. To significantly increase this volume and stay within acceptable limits of computer resources, we use nonequidistant grids. For example, if we apply logarithmic scaling of time:

$$ T = \ln (t) $$

(46)

the heat-transfer equation then becomes:

$$ \Delta A = \mu \sigma e^{-T} \frac{\partial A}{\partial T} $$

(47)

For the new variable $T$, we use an equidistant grid.

In designing spatial grids, we follow the simple principle: a dense equidistant grid is used in the minimal domain that includes the source, the inhomogeneity, and the points of measurement. Outside this domain, the spatial intervals are designed to increase logarithmically. To preserve the accuracy of approximation for the differential operators, we change the FD approximation formulae accordingly.
9. Algorithm verification and application example

Since the algorithm is very complicated, great attention was given to a verification of the program that was named CRAZY. The program was tested using numerous independent methods, both in the development stages and during routine calculations. In the first stage, the finite-difference solution was checked to verify that it satisfied the finite-difference analog of differential equations and boundary conditions (vertical and radial). The convergence of the numerical scheme was then checked; decreasing temporal and spatial grids together while also increasing their density improved the accuracy of the solution. In the next stage, the algorithm was checked against horizontally layered solutions, cylindrically layered solutions, 2-D solutions (mixed boundaries) and, last of all, 3-D solutions (Anderson and Newman, 1985). In the final test, which proved to be the most general, our solution was tested to ascertain whether it satisfied the reciprocity principle. This means that for any 3-D model, reciprocal alteration of transmitter and receiver locations does not change the medium response.

9.1. Horizontal boundaries check

Fig. 5 shows a comparison of calculations performed using CRAZY with the results obtained by specialized code ALEX for a horizontally layered medium (Institute for Geology and Geophysics — IGG, Novosibirsk, former USSR). The medium contains three layers and the source is a vertical magnetic dipole placed on the earth surface. The time in Fig. 5 is normalized by the parameters \( \sigma_1 \) (conductivity), \( \mu \) (magnetic permeability), and \( h_i \) (thickness) of the uppermost layer as:

\[
\tau = \sqrt{8 \pi^2 t / \mu \sigma_1 h_i^2}
\]

The data presented demonstrate that for all existing field components, \( H_z \), \( H_r \), and \( E_\phi \), the error of FD modeling does not exceed 3 percent.

9.2. Vertical boundaries check

In Fig. 6, a cylindrically layered model (borehole with radius \( a \), resistivity \( R_m \), and formation with resistivity \( R_i \)) is shown. The transmitter and receiver (both vertical magnetic dipoles) are positioned at the borehole axis.

![Fig. 5. Algorithm accuracy in a horizontally layered medium.](image-url)
The measured component is \( \frac{\partial B_z}{\partial t} \), where \( z \) indicates the direction parallel to the borehole axis. Two spacings, \( L \), between transmitter and receiver are considered, i.e. \( L/\alpha = 4; 6 \). The time, \( t \), is normalized by the borehole parameters as:

\[
\tau_h = \sqrt{8 \pi^2 \nu R_m / \mu}
\]

The test results for apparent resistivity, \( R_a \), are obtained using a code developed by IGG for the analytical solution. The very close matching of the two solutions proves the high accuracy of CRAZY.

9.3. Mixed boundaries check

The 2-D model results shown in Fig. 7 pertain to a conductive layer, with insulator layers above and below, and a cylinder of finite height embedded in the layer. The model parameters are indicated in Fig. 7. The results obtained using CRAZY are checked against a specific 2-D finite difference code developed by IGG. The time, \( \tau \), is normalized by the parameters of the layer:

\[
\tau_i = \sqrt{8 \pi^2 \nu / \mu \sigma_i}
\]

Apparent resistivity, \( R_a \), is normalized by resistivity of the layer, \( \sigma_i \). Once again, the comparison illustrates the high accuracy of CRAZY.

9.4. Comparison with 3-D modeling (Anderson and Newman, 1985)

We used available results for the model that includes a resistive prism with a square cross-section embedded in a conductive half-space (Fig. 8). The top section of the prism coincides with the bottom of the first layer. The source and receiver are square loops. To approximate the prism, we considered three different circular cylinders: (1) enclosed in the prism, (2) an intermediate case, and (3) enclosing the prism. The best matching was obtained for the first case (Fig. 9). The two cases presented in Fig. 9 differ in the position of the center point between the loops centers \( x = 600 \text{ m}; -1000 \text{ m} \). This model fully represents the 2.5-D situation.
9.5. Reciprocity check

To check reciprocity, two 2.5-D models (Fig. 10) were used. Model A consists of a circular cylinder in a homogeneous half-space. Model B contains a 3-layered half-space and cylinder. The upper-plane boundary crosses the cylinder, the bottom of which is adjacent to the second horizontal boundary. The parts of the cylinder embedded in different layers have different conductivities. The conductivity values are indicated in the icons. The field is excited by a vertical magnetic dipole, and the measured component is $H_z$. The comparison of reciprocal arrays is shown in Fig. 11, where the ratio of reciprocal responses is presented for models A and B. The time is normalized by the parameters of the upper layer:

$$\tau_1 = \sqrt{\frac{8\pi^2 c}{\mu \sigma \gamma h}}$$

Since theoretically for any model reciprocal alternation of the transmitter and receiver locations does not change the medium response, we can evaluate the accuracy of CRAZY through such reciprocal modeling. One can see that it is greater than 98 percent.

9.6. Application example

As an example, we consider the influence on the measurements of the transient logging tool caused by the tool being decentralized from the borehole axis. The tool consists of two horizontal coils (transmitter $Tx$ and

Fig. 7. Example of the algorithm accuracy in an axisymmetric model.
receiver Rx, Fig. 12). The medium consists of the borehole (diameter $B_{m} = 0.2$ m; mud resistivity $R_{m} = 1$ ohm-m), the resistive layer (thickness $h = 0.6$ m; resistivity of virgin formation $R_{l} = 100$ ohm-m), the invasion zone (diameter $D_{m} = 0.4$ m; 1.7 m; 6.1 m; resistivity, $R_{m} = 10$ ohm-m), and the conductive shoulder (resistivity $R_{sh} = 3$ ohm-m). The tool is decentralized away from the borehole axis by 0.05 m. The total length of the tool is 0.3 m. In Fig. 13 we show the ratio between the magnetic field $H_{c}(off)$ for the decentralized condition to the magnetic field $H_{c}(cent.)$ obtained when the tool is centralized. The time, $t$, is normalized by the borehole parameters:

$$\tau(B_h) = \sqrt{8\pi^2 R_m/\mu}$$

The borehole radius is indicated as $B_{m}$ in Fig. 13. As shown by Fig. 13, the tool decentralization manifests itself in the early stages of the transient response $\tau(B_h)/B_{m} < 20$. No difference in the normalized responses for the models with the radii of the invasion zone 1.7 m and 6.1 m is observed. This means that the effect is caused

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**Fig. 10.** Models used for reciprocity checks. Numbers near the icons with different patterns show resistivity values used.

**Fig. 11.** Comparison of results for reciprocal positions of transmitter and receiver.
Fig. 8. Anderson and Newman (1985) prism and cylindrical model used for comparison. Transmitter and receiver are square loops.

Fig. 9. Comparison of 2.5-D modeling with results from Anderson and Newman (1985).
by a different current distribution in the area close to the source, mainly in the conductive borehole. Therefore, the information regarding the virgin formation is practically undistorted.

10. Conclusions

(1) An arbitrary 3-D electromagnetic field generated in the presence of an axially symmetric medium may be described as a series of independent angular modes.

(2) A boundary value problem for any particular angular mode is mathematically two dimensional.

(3) Three independent equations for the three vector potential components are coupled only on the boundaries.

(4) The grid of a finite difference model may cover only conductive areas while the influence of insulators may be described by integral equations spread over the interfaces between conductors and insulators. The integral equations constitute boundary conditions for the parabolic equations that drive the field evolution in the
conductive regions. By combining a finite difference approach with the integral equations, we create a hybrid approach. The major advantage of the hybrid scheme is a significant reduction of computational resources required for numerical implementation.

(5) By using the alternating direction implicit scheme, we substantially simplify linear system solvers and make the overall problem of algorithm and program development much easier.

(6) The application domain of the developed technique covers well logging as well as surface prospecting and exploration. The 2.5-D approximation is especially useful in studying 3-D effects by means of 2-D modeling.

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